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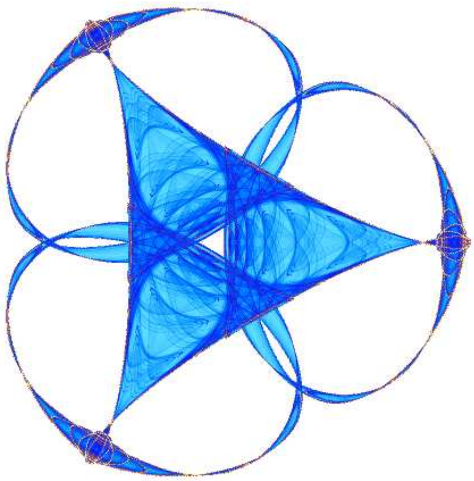

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APPROXIMATE DIRICHLET BOUNDARY CONDITIONS IN THE GENERALIZED FINITE ELEMENT METHOD

IVO BABUŠKA, VICTOR NISTOR, AND NICOLAE TARFULEA

Abstract. We propose a method for treating the Dirichlet boundary conditions in the framework of the Generalized Finite Element Method (GFEM). We use approximate Dirichlet boundary conditions as in [12] and polynomial approximations of the boundary. Our sequence of GFEM-spaces considered, S_μ, $\mu = 1, 2, \ldots$ is such that $S_\mu \not\subset H^1_0(\Omega)$, and hence it does not conform to one of the basic FEM conditions. Let h_μ be the typical size of the elements defining S_μ and let $u \in H^{m+1}(\Omega)$ be the solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, on a smooth, bounded domain Ω. Assume that $\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)}$ for all $v_\mu \in S_\mu$ and $|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^m \|u\|_{H^{m+1}(\Omega)}$, $u \in H^{m+1}(\Omega) \cap H^1_0(\Omega)$, for a suitable $u_I \in S_\mu$. Then we prove that we obtain quasi-optimal rates of convergence for the sequence $u_\mu \in S_\mu$ of GFEM approximations of u, that is, $\|u - u_\mu\|_{H^1(\Omega)} \leq Ch_\mu^m \|f\|_{H^{m-1}(\Omega)}$. Next, we indicate an effective technique for constructing sequences of GFEM-spaces satisfying our conditions using polynomial approximations of the boundary. Finally, we extend our results to the inhomogeneous Dirichlet boundary value problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$.

Contents

1. Introduction

In the past few years, meshless methods for the approximation of solutions of partial differential equations have received increasing attention, especially in the Engineering and Physics communities. The reasons behind the development of such methods are the difficulties associated to the mesh generation, particularly when the geometry of the domain is complicated. As in the case of the usual Finite Element Method, one of the major problems in the implementation of meshless

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methods is the enforcement of Dirichlet boundary conditions. It is the purpose of this paper to address the problem of enforcing Dirichlet boundary conditions in the Generalized Finite Element Method framework.

The classical Rayleigh-Ritz methods for elliptic Dirichlet boundary value problems assume that the trial subspace functions fulfill the boundary conditions. Nevertheless, the construction of such subspaces implies many difficulties in practice when the boundary of the domain is curved. Therefore, several approaches are known for dealing with the Dirichlet boundary conditions. One approach is to modify the variational principles by adding appropriate boundary terms so that there will be no need for the trial subspaces to fulfill any condition at the boundary. See the works of Babuška [2, 3], Bramble and Nitsche [13], and Bramble and Schatz [15, 16], among others, for examples of how this approach works in practice. Another approach (used also in this paper) is to use subspaces with nearly zero boundary conditions. This ideea was first outlined by Nitsche [27] and further studied by Berger, Scott, and Strang [12] and Nitsche [28].

Yet another approach is the Isoparametric Finite Element Method or IFEM with *curved* finite elements along the boundary. See [19] and references therein, or [18], [20, 22, 23, 24, 29, 30], among many others, for more recent work and applications. This approach is typically used in connection with a numerical quadrature scheme computing the coefficients of the resulting linear systems. In the applications of this method, except in special cases (such as when $\overline{\Omega}$ is a polyhedral domain) the interior Ω_h of the union of the finite elements is not equal to Ω, although the boundary of Ω_h is very close to $\partial\Omega$. That is, the approximate solution u_h is sought in a subspace $V_h \subset H^1_0(\Omega_h)$ and so, the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$ is "approximated" by the boundary condition $u_h = 0$ on $\partial\Omega_h$. In fact, u_h is the solution of a variational equation $a_h(u_h, v_h) = (f_h, v_h)_h$ for all $v_h \in V_h$, where $a_h(\cdot, \cdot)$ is a bilinear form which approximates the usual bilinear form defined over $H^1(\Omega_h) \times H^1(\Omega_h)$, and $f_h \in V_h^*$ approximates the linear form $v_h \in V_h \to \int_{\Omega_h} \tilde{f} v_h dx$, where $\tilde{f}$ is an extension of f to the set Ω_h.

Our approach has certain points in common with the isoparametric method just mentioned in the fact that we are using polynomial approximations of the boundary. However, our method does not require non-linear changes of coordinates. Our method combines the approaches in the papers of Berger, Scott, and Strang [12] and Nitsche [28]. Our definition of the discrete solution is as in [12], whereas our assumptions are closer to those of [28]. We have tried to keep our assumptions at a minimum. This is possible using partitions of unity, more precisely the Generalized Finite Element Method or GFEM, a method that originated in the work of Babuška, Caloz, and Osborn [8] and further developed in [6, 9, 10, 25, 26].

Our construction is different from the IFEM in that we do not require complicated non-linear changes of coordinates. Moreover, our method uses non-conforming subspaces of functions and it does not have to deal with extensions over larger domains. It is closely related to [11] which uses GFEM for elliptic Neumann boundary value problems with distributional boundary data. The GFEM is a generalization of the meshless methods which use the idea of partition of unity. This method allows a great flexibility in constructing the trial spaces, permits inclusion of a priori knowledge about the differential equation in the trial spaces, and gives the option of constructing trial spaces of any desired regularity. We mention that the GFEM is also known and used under other names, such as: the method of "clouds," the method of "finite spheres," the "X–finite element method," and others. See the survey by Babuška, Banerjee, and Osborn [5] for further information and references.

Let us now describe the main results of this paper in some detail. Let $\Omega \subset \mathbb{R}^n$ be a *smooth, bounded* domain with boundary $\partial\Omega$. Let $f \in L^2(\Omega)$ and $u \in H^2(\Omega)$ be the unique solution of the Dirichlet problem

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Assume that a sequence $S_\mu \subset H^1(\Omega)$ of test-trial spaces is given and define the discrete solution $u_\mu \in S_\mu$ in the usual way: $B(u_\mu, v_\mu) = (f, v_\mu)$ for all $v_\mu \in S_\mu$ (see Equation (2) below). We do not assume S_μ to satisfy exactly the Dirichlet boundary conditions, that is, we do *not* assume $S_\mu \subset H^1_0(\Omega)$.

Let us fix from now on a natural number $m \in \mathbb{N} = \{1, 2, \dots\}$ that will play, in what follows, the role of the expected *order of approximation*. We shall make the following two basic assumptions:

- *Assumption 1, nearly zero boundary values:* $\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq C h_\mu^m \|v_\mu\|_{H^1(\Omega)}$ for any $v_\mu \in S_\mu$ and
- *Assumption 2, approximability:* for any $u \in H^{j+1}(\Omega) \cap H^1_0(\Omega)$, $0 \leq j \leq m$, there exists $u_I \in S_\mu$ such that $|u - u_I|_{H^1(\Omega)} \leq C h_\mu^j \|u\|_{H^{j+1}(\Omega)}$.

These two assumptions are formulated in more detail in Section 2.

Under Assumptions 1 and 2, our main approximation result (proved in Section 2) is the following

Theorem 1.1. *Let $S_\mu \subset H^1(\Omega)$ be a sequence of finite dimensional subspaces satisfying Assumptions 1 and 2 for a sequence $h_\mu \to 0$ and $1 \leq p \leq m$. Then the (unique) solutions u and u_μ of Equations* (1) *and* (2)*, respectively, with $f \in H^{p-1}(\Omega)$ satisfy*

$$\|u - u_\mu\|_{H^1(\Omega)} \leq C h_\mu^p \|u\|_{H^{p+1}(\Omega)} \leq C h_\mu^p \|f\|_{H^{p-1}(\Omega)},$$

for constants independent of μ and f.

In Assumptions 1 and 2, $h_\mu > 0$ is a sequence that goes to 0. Intuitively, h_μ will play the role of the "typical size" of the elements in S_μ. However, in our abstract setting, we are not assuming that S_μ is constructed in any particular way. Assumptions 1 and 2 are easy to fulfill with a "flat-top" partition of unity and polynomial local approximation spaces. In Sections 3 and 4 we provide examples of spaces S_μ that satisfy Assumptions 1 and 2. In Section 5 we extend our results to the non-homogeneous Dirichlet boundary conditions case $u = g$ on $\partial\Omega$.

In this paper, we shall use the convention that $C > 0$ indicates a generic constant, independent of μ, which may be different each time when used, but is independent of the free variables of the formulas.

2. Approximate Dirichlet boundary conditions

In this section, we give a proof of Theorem 1.1. We begin by fixing the notation and then we prove some preliminary results.

Recall that $\Omega \subset \mathbb{R}^n$ is a *smooth, bounded domain*, fixed throughout this paper. We shall fix in what follows $m \in \mathbb{N} = \{1, 2, \dots\}$, which will play the role of *the order of approximation.* We want to approximate u with functions $u_\mu \in S_\mu$, $\mu \in \mathbb{N}$, where $S_\mu \subset H^1(\Omega)$ is a sequence of finite dimensional subspaces that satisfy the Assumption 1 and 2 formulated next. Our first assumption is:

Assumption 1 (nearly zero boundary values). There exists $C > 0$ such that

$$\|v_\mu\|_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)} \qquad \text{for any } v_\mu \in S_\mu.$$

So S_μ *does not* necessarily consist of functions satisfying the Dirichlet boundary conditions. Let $|u|_{H^1(\Omega)} := [\int_\Omega |\nabla u|^2 dx]^{1/2}$. Our second assumption is:

Assumption 2 (approximability). There exists $C > 0$ such that for any $0 \leq j \leq m$, any $u \in H^{j+1}(\Omega) \cap H_0^1(\Omega)$, and any $\mu \in \mathbb{N}$, there exists $u_I \in S_\mu$ such that

$$|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^j \|u\|_{H^{j+1}(\Omega)}.$$

We now proceed to the proof of Theorem 1.1. We first need some lemmas. Let $Q : H^1(\Omega) \to H_0^1(\Omega)$ be the H^1-orthogonal projection onto the subspace $H_0^1(\Omega) \subset H^1(\Omega)$ of functions satisfying the Dirichlet boundary conditions. Let $S_\mu^0 := Q(S_\mu)$.

Lemma 2.1. *We have that* $\|w - Q(w)\|_{H^1(\Omega)} \leq Ch_\mu^m \|w\|_{H^1(\Omega)}$, $w \in S_\mu$.

Proof. Let us denote by $E : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ a fixed, continuous right inverse of the restriction (or trace map) $H^1(\Omega) \to H^{1/2}(\partial\Omega)$. That is, we have

$$(Ev)|_{\partial\Omega} = v \quad \text{and} \quad \|Ev\|_{H^1(\Omega)} \leq C\|v\|_{H^{1/2}(\partial\Omega)}, \qquad v \in H^{1/2}(\partial\Omega).$$

We can chose such an extension map E since the restriction $H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is continuous and surjective. Consider now $w \in H^1(\Omega)$ arbitrary. Then $w_E := w - E(w|_{\partial\Omega}) \in H_0^1(\Omega)$, and hence the projection property gives

$$\|w - Q(w)\|_{H^1(\Omega)} \leq \|w - w_E\|_{H^1(\Omega)} = \|E(w|_{\partial\Omega})\|_{H^1(\Omega)} \leq C\|w|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)}.$$

If $w \in S_\mu$, Assumption 2 then gives $\|w - Q(w)\|_{H^1(\Omega)} \leq Ch_\mu^m \|w\|_{H^1(\Omega)}$, as desired. ☐

In what follows, we shall need the following classical result [1, 17].

Lemma 2.2. *For* $u \in H^1(\Omega)$ *there is a constant* C *that depends only on* Ω *such that*

$$\|u\|_{H^1(\Omega)}^2 \leq C\big[|u|_{H^1(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2\big].$$

From this lemma we obtain that $|v_\mu|_{H^1(\Omega)}$ and $\|v_\mu\|_{H^1(\Omega)}$ are equivalent norms on S_μ, with equivalence bounds independent of μ:

Lemma 2.3. *There exists* $C > 0$ *such that* $C^{-1}|v_\mu|_{H^1(\Omega)} \leq \|v_\mu\|_{H^1(\Omega)} \leq C|v_\mu|_{H^1(\Omega)}$ *for all* μ *large enough and all* $v_\mu \in S_\mu$.

Proof. From Lemma 2.2, we have

$$\begin{aligned}\|v_\mu\|_{H^1(\Omega)}^2 \leq C\big[|v_\mu|_{H^1(\Omega)}^2 + \|v_\mu\|_{L^2(\partial\Omega)}^2\big] &\leq C\big[|v_\mu|_{H^1(\Omega)}^2 + \|v_\mu\|_{H^{1/2}(\partial\Omega)}^2\big] \\ &\leq C|v_\mu|_{H^1(\Omega)}^2 + Ch_\mu^{2m}\|v_\mu\|_{H^1(\Omega)}^2,\end{aligned}$$

where the last inequality is a consequence of Assumption 1. Therefore, for μ large, h_μ is small enough and we get

$$\|v_\mu\|_{H^1(\Omega)} \leq C(1 - Ch_\mu^{2m})^{-1/2}|v_\mu|_{H^1(\Omega)}$$

which is enough to complete the proof. ☐

Lemma 2.3 allows us now to introduce the discrete solution u_μ of Equation (1) using the standard procedure. Let $B(v,w) := \int_\Omega \nabla v \cdot \nabla w dx$ be the usual bilinear form. For μ large, let us define the discrete solution $u_\mu \in S_\mu$ of the Dirichlet problem (1) by the usual formula

$$B(u_\mu, v_\mu) = \int_\Omega f(x) v_\mu(x) dx, \qquad \text{for all } v_\mu \in S_\mu. \tag{2}$$

Let ν be the outer unit normal to $\partial\Omega$ and dS denote the surface measure on $\partial\Omega$. Similarly, let $w_\mu \in S_\mu$, for μ large, be the solution of the variational problem

$$B(w_\mu, v_\mu) = \int_{\partial\Omega} \partial_\nu u(x) v_\mu(x) dS(x), \qquad \text{for all } v_\mu \in S_\mu, \tag{3}$$

where u is the solution of Equation (1). Note that we need Lemma 2.3 to justify the existence and uniqueness of weak solutions u_μ and w_μ.

Lemma 2.4. *Let u be the solution of the Dirichlet problem* (1) *and let u_μ and w_μ be as in Equations* (2) *and* (3)*. Then $B(u-u_\mu-w_\mu, v_\mu)=0$ for all $v_\mu \in S_\mu$; hence*

$$|u - u_\mu - w_\mu|_{H^1(\Omega)} \le |u - v_\mu|_{H^1(\Omega)} \quad \text{for all } v_\mu \in S_\mu.$$

Proof. This follows from the fact that

$$B(u, v_\mu) = \int_\Omega \nabla u \cdot \nabla v_\mu dx = \int_\Omega f v_\mu dx + \int_{\partial\Omega} (\partial_\nu u) v_\mu dS(x) = B(u_\mu + w_\mu, v_\mu),$$

for all $v_\mu \in S_\mu$. □

We now proceed to estimate u_μ and w_μ.

Lemma 2.5. *For μ large, the solution w_μ of the weak problem* (3) *satisfies*

$$\|w_\mu\|_{H^1(\Omega)} \le C h_\mu^m \|u\|_{H^2(\Omega)}, \tag{4}$$

with C a constant independent of μ and u.

Proof. The relation follows from

$$\begin{aligned}\|w_\mu\|^2_{H^1(\Omega)} \le C|w_\mu|^2_{H^1(\Omega)} = CB(w_\mu, w_\mu) = C\int_{\partial\Omega} \partial_\nu u(x) w_\mu(x) dS(x) \\ \le C\|\partial_\nu u\|_{L^2(\partial\Omega)} \|w_\mu\|_{L^2(\partial\Omega)} \le C\|\partial_\nu u\|_{L^2(\partial\Omega)} \|w_\mu\|_{H^{1/2}(\partial\Omega)} \\ \le C h_\mu^m \|u\|_{H^2(\Omega)} \|w_\mu\|_{H^1(\Omega)}.\end{aligned}$$

Therefore $\|w_\mu\|_{H^1(\Omega)} \le C h_\mu^m \|u\|_{H^2(\Omega)}$, as claimed. □

Lemma 2.6. *For μ large, the solution u_μ of the weak problem* (2) *satisfies*

$$\|u_\mu\|_{H^1(\Omega)} \le C\|u\|_{H^2(\Omega)}, \tag{5}$$

with C a constant independent of μ and u.

Proof. Let us first observe that Lemma 2.3 and then Lemma 2.4 give

$$\begin{aligned}\|u_\mu\|^2_{H^1(\Omega)} \le C|u_\mu|^2_{H^1(\Omega)} = CB(u_\mu, u_\mu) = C\big[B(u, u_\mu) - B(w_\mu, u_\mu)\big] \\ = C\big[B(u, u_\mu) - \langle \partial_\nu u, u_\mu\rangle_{\partial\Omega}\big] \le C\big[|B(u, u_\mu)| + |\langle \partial_\nu u, u_\mu\rangle_{\partial\Omega}|\big] \\ \le C\|u\|_{H^1(\Omega)} \|u_\mu\|_{H^1(\Omega)} + C\|\partial_\nu u\|_{L^2(\partial\Omega)} \|u_\mu\|_{L^2(\partial\Omega)} \\ \le C\|u\|_{H^2(\Omega)} \|u_\mu\|_{H^1(\Omega)} + C h_\mu^m \|u\|_{H^2(\Omega)} \|u_\mu\|_{H^1(\Omega)}.\end{aligned}$$

Now it is easy to see that $\|u_\mu\|_{H^1(\Omega)} \le C\|u\|_{H^2(\Omega)}$. □

Now we are ready to prove Theorem 1.1.

Proof. We shall assume $p = m$, for simplicity. The proof in general is exactly the same. Lemma 2.4 and the projection property, together with Lemma 2.5, give

$$\begin{aligned}(6)\quad |u-u_\mu|_{H^1(\Omega)} &\le |u-u_\mu-w_\mu|_{H^1(\Omega)} + |w_\mu|_{H^1(\Omega)} \\ &\le |u-u_I|_{H^1(\Omega)} + Ch_\mu^m\|u\|_{H^2(\Omega)} \le Ch_\mu^m\|u\|_{H^{m+1}(\Omega)},\end{aligned}$$

where for the last line we also used the approximation property (Assumption 2).

The estimate in the H^1-norm is obtained from Lemma 2.2, Equation (6), Assumption 1, and Lemma 2.5 as follows

$$\begin{aligned}\|u-u_\mu\|_{H^1(\Omega)} &\le C\big[|u-u_\mu|_{H^1(\Omega)} + \|u_\mu\|_{L^2(\partial\Omega)}\big] \\ &\le Ch_\mu^m\|u\|_{H^{m+1}(\Omega)} + Ch_\mu^m\|u_\mu\|_{H^1(\Omega)} \le Ch_\mu^m\|u\|_{H^{m+1}(\Omega)}.\end{aligned}$$

The proof is now complete. $\square$

In view of some further applications, we now include an error estimate in a "negative order" Sobolev norm. We let $H^{-l}(\Omega)$ to be the dual of $H^l(\Omega)$ with pivot $L^2(\Omega)$. Since Ω is a smooth domain, $H^{-l}(\Omega)$ can also be described as the closure of $\mathcal{C}^\infty(\Omega)$ in the norm

$$\|u\|_{H^{-l}(\Omega)} = \sup_{\phi\neq 0}\frac{|(u,\phi)_{L^2(\Omega)}|}{\|\phi\|_{H^l(\Omega)}} \tag{7}$$

(Note that, in several other papers, $H^{-l}(\Omega)$ denotes the dual of $H_0^l(\Omega)$.)

Theorem 2.7. *Let $0 \le l$, $1 \le p$, and $l+p+1 \le m$. Then, under the assumptions of Theorem 1.1, the solutions u and u_μ of Equation* (1) *and Equation* (2)*, respectively, satisfy*

$$\|u-u_\mu\|_{H^{-l}(\Omega)} \le Ch_\mu^{l+p+1}\|u\|_{H^{p+1}(\Omega)},$$

for a constant $C > 0$ independent of μ and $f \in H^{p-1}(\Omega)$.

Proof. The proof of this theorem is an adaptation of the usual Nitsche-Aubin trick. Indeed, let us denote by $F \in H^{l+2}(\Omega)$ the unique solution of the equation $-\Delta F = \phi$, $F = 0$ on $\partial\Omega$, for $\phi \in H^l(\Omega)$ arbitrary, non-zero. Then there exists a constant $C > O$, independent of ϕ, such that $\|F\|_{H^{l+2}(\Omega)} \le C\|\phi\|_{H^l(\Omega)}$. By Assumption 2, there exists $F_I \in S_\mu$ such that $|F-F_I|_{H^1(\Omega)} \le Ch_\mu^{l+1}\|F\|_{H^{l+2}(\Omega)}$. An easy observation, which will be used later, is that

$$|F_I|_{H^1(\Omega)} \le Ch_\mu^{l+1}\|F\|_{H^{l+2}(\Omega)} + |F|_{H^1(\Omega)} \le C\|\phi\|_{H^l(\Omega)}. \tag{8}$$

Using Lemmas 2.1, 2.5, and 2.6, together with Assumptions 1–2 and (8), we obtain

$$\begin{aligned}
\|u-u_\mu\|_{H^{-l}(\Omega)} &= \sup_{\phi\neq 0}\frac{|(u-u_\mu,\phi)_{L^2(\Omega)}|}{\|\phi\|_{H^l(\Omega)}} = \sup_{\phi\neq 0}\frac{\left|B(u-u_\mu,F)+\int_{\partial\Omega}u_\mu\partial_\nu F dS\right|}{\|\phi\|_{H^l(\Omega)}}\\
&\leq \sup_{\phi\neq 0}\frac{|B(u-u_\mu,F-F_I)|}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi\neq 0}\frac{|B(w_\mu,F_I)|}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi\neq 0}\frac{\left|\int_{\partial\Omega}u_\mu\partial_\nu F dS\right|}{\|\phi\|_{H^l(\Omega)}}\\
&\leq \sup_{\phi\neq 0}\frac{|u-u_\mu|_{H^1(\Omega)}|F-F_I|_{H^1(\Omega)}}{\|\phi\|_{H^l(\Omega)}} + \sup_{\phi\neq 0}\frac{|w_\mu|_{H^1(\Omega)}|F_I|_{H^1(\Omega)}}{\|\phi\|_{H^l(\Omega)}}\\
&\quad + \sup_{\phi\neq 0}\frac{\|u_\mu\|_{L^2(\partial\Omega)}\|\partial_\nu F\|_{L^2(\partial\Omega)}}{\|\phi\|_{H^l(\Omega)}}\\
&\leq Ch_\mu^{p+l+1}\|u\|_{H^{p+1}(\Omega)} + Ch_\mu^m\|u\|_{H^2(\Omega)} + Ch_\mu^m\|u\|_{H^2(\Omega)}\\
&\leq Ch_\mu^{p+l+1}\|u\|_{H^{p+1}(\Omega)},
\end{aligned}$$

by the assumption that $p+l+1\leq m$. □

3. The Generalized Finite Element Method

Our goal is to construct a sequence S_μ, $\mu = 1,2,\ldots$, of Generalized Finite Element spaces that satisfy the two assumptions of the previous section. To this end, we shall introduce a sequence of Generalized Finite Element spaces that satisfy certain conditions (Conditions A(h_μ), B, C, and D). In the following sections we shall prove that these conditions imply Assumptions 1 and 2.

We begin by recalling a few basic facts about the Generalized Finite Element Method [5, 10, 26]. This method is quite convenient when one needs test or trial spaces with high regularity.

3.1. **Basic facts.** Let $k\in\mathbb{Z}_+$. We shall denote as usual

$$|u|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|=k}\|\partial^\alpha u\|_{L^\infty(\Omega)},\quad \|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|\leq k}\|\partial^\alpha u\|_{L^\infty(\Omega)},$$

$W^{k,\infty}(\Omega) := \{u, \|u\|_{W^{k,\infty}(\Omega)} < \infty\}$, and $\|\nabla\omega\|_{W^{k,\infty}(\Omega)} := \sum_j\|\partial_j\omega\|_{W^{k,\infty}(\Omega)}$. In particular, $|u|_{W^{0,\infty}(\Omega)} = \|u\|_{W^{0,\infty}(\Omega)} = \|u\|_{L^\infty(\Omega)}$.

We shall need the following slight generalization of a definition from [10, 26]:

Definition 3.1. Let $\Omega\subset\mathbb{R}^n$ be an open set and $\{\omega_j\}_{j=1}^N$ be an open cover of Ω such that any $x\in\Omega$ belongs to at most κ of the sets ω_j. Also, let $\{\phi_j\}$ be a partition of unity consisting of $W^{m,\infty}(\Omega)$ functions and subordinated to the covering $\{\omega_j\}$ (*i.e.*, $\operatorname{supp}\phi_j\subset\overline{\omega}_j$). If

$$\|\partial^\alpha\phi_j\|_{L^\infty(\Omega)}\leq C_k/(\operatorname{diam}\omega_j)^k,\quad k=|\alpha|\leq m, \tag{9}$$

for any $j=1,\ldots,N$, then $\{\phi_j\}$ is called a $(\kappa, C_0, C_1,\ldots,C_m)$ *partition of unity.*

Assume also that we are given linear subspaces $\Psi_j\subset H^m(\omega_j)$, $j=1,2,\ldots,N$. The spaces Ψ_j will be called *local approximation spaces* and will be used to define the space

$$S = S_{GFEM} := \Big\{\sum_{j=1}^N\phi_j v_j,\ v_j\in\Psi_j\Big\}\subset H^m(\Omega), \tag{10}$$

which will be called the *GFEM–space*. The set $\{\omega_j, \phi_j, \Psi_j\}$ will be called the *set of data defining the GFEM–space* S. A basic approximation property of the GFEM–spaces is the following Theorem from [10].

Theorem 3.2 (Babuška-Melenk). *We shall use the notations and definitions of Definition 3.1 and after. Let $\{\phi_j\}$ be a (κ, C_0, C_1) partition of unity. Also, let $v_j \in \Psi_j \subset H^1(\omega_j)$, $u_{ap} := \sum_j \phi_j v_j \in S$, and $d_j = \operatorname{diam} \omega_j$, the diameter of ω_j. Then*

$$\|u - u_{ap}\|^2_{L^2(\Omega)} \leq \kappa C_0^2 \sum_j \|u - v_j\|^2_{L^2(\omega_j)} \quad \textit{and}$$

$$\|\nabla(u - u_{ap})\|^2_{L^2(\Omega)} \leq 2\kappa \sum_j \Big(\frac{C_1^2 \|u - v_j\|^2_{L^2(\omega_j)}}{(d_j)^2} + C_0^2 \|\nabla(u - v_j)\|^2_{L^2(\omega_j)} \Big). \tag{11}$$

3.2. **Conditions on GFEM data defining S_μ.** Recall that ω is *star-shaped* with respect to $\omega^* \subset \omega$ if, for every $x \in \omega$ and every $y \in \omega^*$, the segment with end points x and y is completely contained in ω. Also, recall that we have fixed an integer m that plays the role of the order of approximation. Let $\{\omega_j, \phi_j, \Psi_j\}_{j=1}^N$ be a *single*, fixed data defining a GFEM–space S, as in the previous subsection, and let $\Sigma := \{\omega_j, \phi_j, \Psi_j, \omega_j^*\}$, where ω_j is star-shaped with respect to $\omega_j^* \subset \omega_j$. We shall need, in fact, to consider a sequence of such data

$$\Sigma_\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}_{j=1}^{N_\mu}, \quad \mu \in \mathbb{N}, \tag{12}$$

defining GFEM–spaces S_μ

$$S_\mu := \Big\{ \sum_{j=1}^{N_\mu} \phi_j^\mu v_j,\ v_j \in \Psi_j^\mu \Big\} \subset H^m(\Omega), \tag{13}$$

such that there exist constants A, C_j, σ, and κ and a sequence $h_\mu \to 0$, as $\mu \to \infty$, for which Σ_μ satisfies Conditions A(h_μ), B, C, and D below for $\mu \in \mathbb{N}$.

Condition A(h_μ). *We have that $\Omega = \cup_{j=1}^{N_\mu} \omega_j^\mu$ and for each $j = 1, 2, \ldots, N_\mu$, the set ω_j^μ is open of diameter $d_j^\mu \leq h_\mu \leq 1$ and $\omega_j^{*\mu} \subset \omega_j^\mu$ is an open ball of diameter $\geq \sigma d_j^\mu$ such that ω_j^μ is star-shaped with respect to $\omega_j^{*\mu}$.*

Notice that we only assume the open covering $\{\omega_j^\mu\}$ to be *nondegenerate*, a weaker condition than *quasi-uniformity* (see [17], Section 4.4, for definitions and more information on these notions).

Condition B. *The family $\{\phi_j^\mu\}_{j=1}^{N_\mu}$ is a $(\kappa, C_0, C_1, \ldots, C_m)$ partition of unity.*

The following condition defines the local approximation spaces Ψ_j^μ. To formulate this condition, let us choose $x_j \in \overline{\omega_j^\mu} \cap \partial\Omega$, if the intersection is not empty. We can assume that linear coordinates have been chosen such that $x_j = 0$ and the tangent space to $\partial\Omega$ at x_j is $\{x_n = 0\} = \mathbb{R}^{n-1}$. For h_μ small, we can assume that $\overline{\omega_j^\mu} \cap \partial\Omega$ is contained in the graph of a smooth function $g_j^\mu : \mathbb{R}^{n-1} \to \mathbb{R}$. If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then we shall denote $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, so that $x = (x', x_n)$. Let $q_j^\mu : \mathbb{R}^{n-1} \to \mathbb{R}$ be a polynomial of order m such that

$$\begin{aligned} |g_j^\mu(x') - q_j^\mu(x')| &\leq C(d_j^\mu)^{m+1} \quad \text{and} \\ |\nabla g_j^\mu(x') - \nabla q_j^\mu(x')| &\leq C(d_j^\mu)^m \quad \text{for all } (x', x_n) \in \omega_j^\mu. \end{aligned} \tag{14}$$

This condition is satisfied, for instance, if $\partial^\alpha g_j^\mu(0) = \partial^\alpha q_j^\mu(0)$, for all $|\alpha| \le m$. In this case, the m-degree polynomial $q_j^\mu : \mathbb{R}^{n-1} \to \mathbb{R}$ is uniquely defined by the afore mentioned requirement.

Next, denote by $\tilde q_j^\mu : \mathbb{R}^n \to \mathbb{R}^n$ the bijective map

$$\tilde q_j^\mu(x) = \tilde q_j^\mu(x', x_n) = (x', x_n + q_j^\mu(x')). \tag{15}$$

Let us denote by $\mathcal{P}_k$ the space of polynomials of order at most k in n variables.

Condition C. *We have* $\Psi_j^\mu = \mathcal{P}_m$ *if* $\overline{\omega_j^\mu} \cap \partial\Omega = \emptyset$ *and, otherwise,*

$$\Psi_j^\mu = \{p \circ (\tilde q_j^\mu)^{-1},\ p \in \mathcal{P}_m, \ \text{ such that } \ p(x', 0) = 0\},$$

where q_j^μ are polynomials satisfying Equation (14) with a constant C independent of j and μ.

An equivalent form of the condition "$p \in \mathcal{P}_m,\ p(x', 0) = 0$" is "$p = x_n p_1$, $p_1 \in \mathcal{P}_{m-1}$," because any polynomial vanishing on the hyperplane $\{x_n = 0\}$ is a multiple of x_n. Since $(\tilde q_j^\mu)^{-1}(x', x_n) = (x', x_n - q_j^\mu(x'))$, we obtain $p(x', x_n) = (x_n - q_j^\mu(x')) p_1 \circ (\tilde q_j^\mu)^{-1}$.

Condition D. *We have* $\phi_j^\mu = 1$ *on* $\omega_j^{*\mu}$ *for all* $j = 1, \dots, N_\mu$ *for which* $\overline{\omega_j^\mu} \cap \partial\Omega \neq \emptyset$.

The constants C_j, σ, and κ will be called *structural constants*. Note that we must have $N_\mu \to \infty$ as $\mu \to \infty$.

The above assumptions are slightly weaker than the ones introduced in [11]. For instance, Condition C implies the following propety (which is similar to Condition C in [11])

For any $w \in \Psi_j^\mu$, *any* $0 \le l \le m+1$, *and any ball* $\omega^* \subset \omega_j^\mu$ *of diameter* $\ge \sigma d_j^\mu$.

$$\|w\|_{H^l(\omega_j^\mu)} \le C \|w\|_{H^l(\omega^*)}. \tag{16}$$

For further applications, we shall also need a variant of the spaces S_μ in which *no boundary conditions are imposed*. Recall the functions q_j^μ used to define the spaces Ψ_j^μ. Let $\tilde\Psi_j^\mu = \Psi_j^\mu$ if ω_j does not touch the boundary $\partial\Omega$ and $\tilde\Psi_j^\mu = \{p \circ (\tilde q_j^\mu)^{-1},\ p \in \mathcal{P}_m\}$ otherwise, (the difference is that we no longer require p to vanish when $x_n = 0$). We then define

$$\tilde S_\mu := \Big\{ \sum_{j=1}^{N_\mu} \phi_j^\mu v_j,\ v_j \in \tilde\Psi_j^\mu \Big\} \subset H^m(\Omega). \tag{17}$$

We shall also need the following standard lemma, a proof of which, for $s \in \mathbb{Z}_+$, can be found in [11]. For $s \ge 0$ it is proved by interpolation.

Lemma 3.3. *Let* ψ_j *be measurable functions defined on an open set* W *and* $s \ge 0$. *Assume that there exists an integer* κ *such that a point* $x \in W$ *can belong to no more than* κ *of the sets* $\mathrm{supp}(\psi_j)$. *Let* $f = \sum_j \psi_j$. *Then there exists a constant* $C > 0$, *depending only on* κ, *such that* $\|f\|_{H^s(W)}^2 \le C \sum_j \|\psi_j\|_{H^s(W)}^2$.

Recall that d_j^μ denotes the diameter of ω_j^μ. Let us observe that Condition A(h_μ) implies the following inverse inequality.

Lemma 3.4. *There exists $C > 0$, depending only on σ, such that*

$$\|p\|_{H^s(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s}\|p\|_{H^r(\omega_j^\mu)}, \tag{18}$$

for all $0 \leq r \leq s \leq m$, all j, all μ, and all polynomials p of order m.

Proof. The proof of this lemma is inspired from the proof of (4.5.3) Lemma of [17].

Consider μ and $1 \leq j \leq N_\mu$ arbitrary, but fixed for the moment. Let

$$\hat{\omega}_j^\mu := \{\frac{1}{d_j^\mu}(x - x_j^\mu),\, x \in \omega_j^\mu\}, \quad \hat{\omega}_j^{*\mu} := \{\frac{1}{d_j^\mu}(x - x_j^\mu),\, x \in \omega_j^{*\mu}\},$$

where x_j^μ is the center of the ball $\omega_j^{*\mu}$.

If $p \in \mathcal{P}_m$ is a polynomial of order m, then $\hat{p}$ is defined by $\hat{p}(\hat{x}) := p(d_j^\mu \hat{x} + x_j^\mu)$ for all $\hat{x}$. Observe that the set $\hat{\mathcal{P}}_m := \{\hat{p} : p \in \mathcal{P}_m\}$ is nothing but the set of all m-degree polynomials in $\hat{x}$. Clearly,

$$|\hat{p}|_{H^k(\hat{\omega}_j^\mu)} = (d_j^\mu)^{k-n/2}|p|_{H^k(\omega_j^\mu)}, \quad \text{for } 0 \leq k \leq m. \tag{19}$$

We first prove (18) for the case $r = 0$. Since $\hat{\mathcal{P}}_m$ is finite dimensional, we have by the equivalence of norms on the unit ball $B(0,1)$ that

$$\|\hat{p}\|_{H^k(B(0,1))} \leq C\|\hat{p}\|_{L^2(B(0,1))}, \quad \text{for any } 0 \leq k \leq m, \tag{20}$$

where $C > 0$ is a constant that does not depend on k, j, and μ. From Condition A(h_μ), we obtain that

$$\|\hat{p}\|_{L^2(B(0,1))} \leq C\|\hat{p}\|_{L^2(\hat{\omega}_j^{*\mu})} \tag{21}$$

where $C > 0$ depends only on the structural constant σ. From (20) and (21), it is clear that

$$\|\hat{p}\|_{H^k(\hat{\omega}_j^\mu)} \leq C\|\hat{p}\|_{L^2(\hat{\omega}_j^\mu)} \quad \forall \hat{p} \in \hat{\mathcal{P}}_m,$$

where $C > 0$ depends only on σ. Therefore, (19) implies

$$|p|_{H^k(\omega_j^\mu)}(d_j^\mu)^{k-n/2} \leq C\|p\|_{L^2(\omega_j^\mu)}(d_j^\mu)^{-n/2} \quad \text{for } 0 \leq k \leq s,$$

from which we deduce that

$$|p|_{H^k(\omega_j^\mu)} \leq C(d_j^\mu)^{-k}\|p\|_{L^2(\omega_j^\mu)} \quad \text{for } 0 \leq k \leq s.$$

Since $d_j^\mu \leq h_\mu \leq 1$, we have

$$\|p\|_{H^s(\omega_j^\mu)} \leq C(d_j^\mu)^{-s}\|p\|_{L^2(\omega_j^\mu)}, \tag{22}$$

which is just (18) for $r = 0$.

Let us now analyse the general case $0 \leq r \leq s \leq m$. For $|\alpha| = k$, with $s - r \leq k \leq s$, $D^\alpha p = D^\beta D^\gamma p$ for $|\beta| = s - r$ and $|\gamma| = k + r - s$. Therefore,

$$\begin{aligned}\|D^\alpha p\|_{L^2(\omega_j^\mu)} &\leq \|D^\gamma p\|_{H^{s-r}(\omega_j^\mu)} \\ &\leq C(d_j^\mu)^{r-s}\|D^\gamma p\|_{L^2(\omega_j^\mu)} \quad (\text{ by } (22)) \\ &\leq C(d_j^\mu)^{r-s}|p|_{H^{k+r-s}(\omega_j^\mu)}.\end{aligned}$$

Since

$$|p|_{H^k(\omega_j^\mu)} := \sum_{|\alpha|=k}\|D^\alpha p\|_{L^2(\omega_j^\mu)},$$

we obtain that

$$|p|_{H^k(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s}|p|_{H^{k+r-s}(\omega_j^\mu)} \quad \text{for } s - r \leq k \leq s.$$

This implies that

$$|p|_{H^k(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s}\|p\|_{H^r(\omega_j^\mu)} \quad \text{for } s-r \leq k \leq s. \tag{23}$$

From (22), we also have

$$\|p\|_{H^{s-r}(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s}\|p\|_{L^2(\omega_j^\mu)} \leq C(d_j^\mu)^{r-s}\|p\|_{H^r(\omega_j^\mu)}. \tag{24}$$

Combining (23) and (24) gives (18) and this ends the proof of the lemma. □

4. Properties of the spaces S_μ

In this section, we establish some properties of the GFEM spaces S_μ, $\mu \in \mathbb{N}$, defined in Equation (13) using the data $\Sigma_\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}_{j=1}^{N_\mu}$ satisfying conditions A(h_μ), B, C, and D introduced in the previous section for $h_\mu \to 0$. The main result is that the sequence S_μ satisfies Assumptions 1 and 2 of the first section.

Hereafter, for simplicity, we will omit the index μ whenever its appearance is implicit.

Let us fix j such that $\overline{\omega}_j \cap \partial\Omega$ is not empty. Recall the functions $g_j, q_j : \mathbb{R}^{n-1} \to \mathbb{R}$ defined in the previous section. So, for h small, $\overline{\omega}_j \cap \partial\Omega$ is contained in $\{(x', g_j(x'))\}$, the graph of the smooth function $g_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (this may require a preliminary rotation, which is not included in the notation, however, for the sake of simplicity). Let $\tilde{q}_j : \mathbb{R}^n \to \mathbb{R}^n$ be the bijective map defined by Equation (15). Similarly, let

$$\tilde{g}_j(x) = \tilde{g}_j(x', x_n) = (x', x_n + g_j(x')). \tag{25}$$

Then $\tilde{g}_j$ maps $\mathbb{R}^{n-1}$ to a surface containing $\overline{\omega}_j \cap \partial\Omega$. We have $\tilde{g}_j^{-1}(x) = (x', x_n - g_j(x'))$ and $\tilde{q}_j^{-1}(x) = (x', x_n - q_j(x'))$.

We shall need the following estimate.

Lemma 4.1. *For any polynomial p of order m, we have*

$$\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \leq C d_j^{m+1}\|p\|_{H^1(\omega_j)} \quad \textit{and}$$
$$\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \leq C d_j^{m}\|p\|_{H^1(\omega_j)},$$

where C is a constant independent of p, μ, and j.

Proof. By Taylor's expansion theorem in the x_n variable, we have

$$p \circ \tilde{g}_j^{-1}(x', x_n) = p(x', x_n - g_j(x')) = p(x', x_n) - g_j(x')\partial_n p(x', x_n) + \dots + (-1)^k \frac{g_j(x')^k}{k!}\partial_n^k p(x', x_n) + \dots + (-1)^m \frac{g_j(x')^m}{m!}\partial_n^m p(x', x_n)$$

and

$$p \circ \tilde{q}_j^{-1}(x', x_n) = p(x', x_n - q_j(x')) = p(x', x_n) - q_j(x')\partial_n p(x', x_n) + \dots + (-1)^k \frac{q_j(x')^k}{k!}\partial_n^k p(x', x_n) + \dots + (-1)^m \frac{q_j(x')^m}{m!}\partial_n^m p(x', x_n).$$

Then,

$$|p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)| = |p(x', x_n - g_j(x')) - p(x', x_n - q_j(x'))|$$
$$\leq |g_j(x') - q_j(x')| \cdot |\partial_n p(x', x_n)| + \dots + \left|\frac{g_j(x')^k - q_j(x')^k}{k!}\right| \cdot |\partial_n^k p(x', x_n)| + \dots + \left|\frac{g_j(x')^m - q_j(x')^m}{m!}\right| \cdot |\partial_n^m p(x', x_n)|.$$

From this and the Cauchy–Schwartz inequality, we obtain

$$
\begin{aligned}
&|p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 = |p(x', x_n - g_j(x')) - p(x', x_n - q_j(x'))|^2 \\
&\leq m[(g_j(x') - q_j(x'))^2 \partial_n p(x', x_n)^2 + \ldots + \frac{(g_j(x')^k - q_j(x')^k)^2}{(k!)^2} \partial_n^k p(x', x_n)^2 + \ldots \\
&\qquad + \frac{(g_j(x')^m - q_j(x')^m)^2}{(m!)^2} \partial_n^m p(x', x_n)^2].
\end{aligned}
$$

Notice that $|g_j(x')| \leq d_j$, for all $(x', x_n) \in \omega_j$, and because $q_j(x') = g_j(x') + O(d_j^{m+1})$, for all $(x', x_n) \in \omega_j$, we have

$$
\big(g_j(x')^k - q_j(x')^k\big)^2 = [g_j^k(x') - (g_j(x') + O(d_j^{m+1}))^k]^2 \leq C d_j^{2(m+k)}, \text{ for } k = 1, \ldots, m,
$$

which in turn implies that

$$
\begin{aligned}
|p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 &\leq C d_j^{2(m+1)} [\partial_n p(x', x_n)^2 + d_j^2 \partial_n^2 p(x', x_n)^2 + \ldots \\
&\quad + d_j^{2(k-1)} \partial_n^k p(x', x_n)^2 + \ldots + d_j^{2(m-1)} \partial_n^m p(x', x_n)^2].
\end{aligned}
$$

By using the inverse inequality $d_j^{k-1} \|p\|_{H^k(\omega_j)} \leq C \|p\|_{H^1(\omega_j)}$, we get

$$
\begin{aligned}
&\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)}^2 = \int_{\omega_j} |p \circ \tilde{g}_j^{-1}(x', x_n) - p \circ \tilde{q}_j^{-1}(x', x_n)|^2 dx' dx_n \\
&\leq C d_j^{2(m+1)} \int_{\omega_j} [\partial_n p(x', x_n)^2 + d_j^2 \partial_n^2 p(x', x_n)^2 + \ldots + d_j^{2(k-1)} \partial_n^k p(x', x_n)^2 + \ldots \\
&\qquad + d_j^{2(m-1)} \partial_n^m p(x', x_n)^2] dx' dx_n \\
&\leq C d_j^{2(m+1)} [\|p\|_{H^1(\omega_j)}^2 + \ldots + d_j^{2(k-1)} \|p\|_{H^k(\omega_j)}^2 + \ldots + d_j^{2(m-1)} \|p\|_{H^m(\omega_j)}^2] \\
&\qquad \leq C d_j^{2(m+1)} \|p\|_{H^1(\omega_j)}^2,
\end{aligned}
$$

and this completes the proof of $\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \leq C d_j^{m+1} \|p\|_{H^1(\omega_j)}$.

The proof of $\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)}$ is reduced to the previous inequality as follows. First, from the inverse inequality $d_j \|p\|_{H^1(\omega_j)} \leq C \|p\|_{L^2(\omega_j)}$, we obtain

$$
\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \leq C d_j^m \|p\|_{L^2(\omega_j)}. \tag{26}
$$

It is then enough to show that

$$
\|\partial_k (p \circ \tilde{g}_j^{-1}) - \partial_k (p \circ \tilde{q}_j^{-1})\|_{L^2(\omega_j)} \leq C d_j^m \|p\|_{H^1(\omega_j)}, \tag{27}
$$

for all $k = 1, 2, \ldots, n$.

The case $k = n$ is easier, so we shall treat only the case when $1 \leq k \leq n - 1$. A Taylor expansion with respect to the x_n-variable gives

$$
\begin{aligned}
\partial_k (p \circ \tilde{g}_j^{-1})(x', x_n) &= \partial_k (p(x', x_n - g_j(x'))) \\
&= (\partial_k p)(x', x_n - g_j(x')) - \partial_k g_j(x') (\partial_n p)(x', x_n - g_j(x'))
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_k (p \circ \tilde{q}_j^{-1})(x', x_n) &= \partial_k (p(x', x_n - q_j(x'))) \\
&= (\partial_k p)(x', x_n - q_j(x')) - \partial_k q_j(x') (\partial_n p)(x', x_n - q_j(x'))
\end{aligned}
$$

Equation (27) then follows from Equation (26) and from the estimates $q_j(x') = g_j(x') + O(d_j^{m+1})$, $\partial_k q_j(x') = \partial_k g_j(x') + O(d_j^m)$ and $|g_j(x')| \leq d_j$ for $(x', x_n) \in \omega_j$ (see Equation (14) and Condition C). □

Remark 4.2. Let us observe that Condition A(h_μ) was used implicitly in the proof of Lemma 4.1 when we used the inverse estimates $d_j^{k-1}\|p\|_{H^k(\omega_j)} \leq C\|p\|_{H^1(\omega_j)}$.

Remark 4.3. If (14) is replaced by the more restrictive condition $|\partial^\alpha(g_j - q_j)| \leq Cd_j^{m+1-|\alpha|}$, for all $|\alpha| \leq m+1$, then the result of the above lemma can be extended as follows: For any polynomial p of order m, we have

$$\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^s(\omega_j)} \leq Cd_j^{m+1-s}\|p\|_{H^1(\omega_j)},\ s = 0, \ldots, m+1,$$

where C is a constant independent of p, μ, j, and s.

Corollary 4.4. *Let $p \in \mathcal{P}_m$, then*

$$\|\phi_j\big(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\big)\|_{H^1(\omega_j)} \leq Cd_j^m\|p\|_{H^1(\omega_j)}.$$

If $p \in \mathcal{P}_m$ also vanishes on $\{x_n = 0\}$ then we have

$$\|\phi_j(p \circ \tilde{q}_j^{-1})\|_{H^{1/2}(\partial\Omega)} \leq Cd_j^m\|p\|_{H^1(\omega_j^*)}.$$

Here C is a constant independent of p, μ, and j.

Proof. Using Lemma 4.1 and Assumption B, we obtain

$$\begin{aligned}\|\phi_j\big(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\big)\|_{H^1(\omega_j)} &\leq \|\phi_j\|_{L^\infty(\omega_j)}\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{H^1(\omega_j)} \\ &\quad + \big(\Sigma_{i=1}^n\|\partial_i\phi_j\|_{L^\infty(\omega_j)}\big)\|p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\|_{L^2(\omega_j)} \\ &\leq Cd_j^m\|p\|_{H^1(\omega_j)} + Cd_j^{-1}d_j^{m+1}\|p\|_{H^1(\omega_j)} \leq Cd_j^m\|p\|_{H^1(\omega_j)}.\end{aligned}$$

The last part follows from the first part of this corollary, which we have already proved, and from the fact that $\phi_j(p \circ \tilde{g}_j^{-1}) = 0$ on $\partial\Omega$. Indeed,

$$\begin{aligned}\|\phi_j(p \circ \tilde{q}_j^{-1})\|_{H^{1/2}(\partial\Omega)} &= \|\phi_j\big(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\big)\|_{H^{1/2}(\partial\Omega)} \\ &\leq C\|\phi_j\big(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\big)\|_{H^1(\Omega)} = C\|\phi_j\big(p \circ \tilde{g}_j^{-1} - p \circ \tilde{q}_j^{-1}\big)\|_{H^1(\omega_j)} \\ &\leq Cd_j^m\|p\|_{H^1(\omega_j)} \leq Cd_j^m\|p\|_{H^1(\omega_j^*)}\end{aligned}$$

The proof is now complete. □

We are ready now to prove that Assumption 1 is satisfied by the sequence of $GFEM$–spaces S_μ introduced in Subsection 3.2.

Proposition 4.5. *Let S_μ be the sequence of GFEM–spaces defined by data Σ_μ (Equation* (12)*) satisfying conditions A(h_μ), B, C, and D. Then the sequence S_μ satisfies Assumption 1.*

Proof. Let $w_j \in \Psi_j^\mu$ and $w = \sum \phi_j w_j \in S_\mu$. Since we are interested in evaluating w at $\partial\Omega$, we can assume that only the terms corresponding to j for which $\overline{\omega}_j \cap \partial\Omega \neq \emptyset$ appear in the sum. Then $w_j = p_j \circ \tilde{q}_j^{-1}$, for some polynomials $p_j \in \mathcal{P}_m$ vanishing on $\{x_n = 0\}$. Hence Lemma 3.3 and Corollary 4.4 give

$$\begin{aligned}\|w\|^2_{H^{1/2}(\partial\Omega)} &\leq C\sum_j \|\phi_j w_j\|^2_{H^{1/2}(\partial\Omega)} \leq C\sum_j \|\phi_j(p_j \circ \tilde{q}_j^{-1})\|^2_{H^{1/2}(\partial\Omega)} \\ &\leq C\sum_j d_j^{2m}\|p_j\|^2_{H^1(\omega_j^*)} \leq Ch_\mu^{2m}\sum_j \|p_j\|^2_{H^1(\omega_j^*)} \leq Ch_\mu^{2m}\sum_j \|w_j\|^2_{H^1(\omega_j^*)}.\end{aligned}$$

By Condition D, $\sum_j \|w_j\|^2_{H^1(\omega_j^*)} = \|w\|^2_{H^1(\cup\omega_j^*)}$. Therefore,

$$\|w\|^2_{H^{1/2}(\partial\Omega)} \leq Ch_\mu^{2m}\|w\|^2_{H^1(\cup\omega_j^*)} \leq Ch_\mu^{2m}\|w\|^2_{H^1(\Omega)}.$$

Assumption 1 is hence satisfied by taking square roots. □

Remark 4.6. Condition D is only needed in the proof of Proposition 4.5. Although one can prove that

$$\sum_j \|w_j\|^2_{H^1(\omega_j^*)} \leq C\|w\|^2_{H^1(\Omega)} \tag{28}$$

(by using norm equivalence in finite dimensional spaces), one can not bypass Condition D because the constant C in (28) depends on μ. To remove this dependence, one would have to impose additional and/or different conditions on the partition of unity.

The proof that the sequence S_μ also satisfies Assumption 2 is also based on the above lemma and on the following result. Recall that the local approximation spaces Ψ_j and $\tilde\Psi_j^\mu$ were defined in Subsection 3.2.

Lemma 4.7. *Let $u \in H^{m+1}(\omega_j)$. Then there exists a polynomial $w \in \tilde\Psi_j^\mu$ such that $\|u-w\|_{H^1(\omega_j)} \leq Cd_j^m\|u\|_{H^{m+1}(\omega_j)}$ and $\|u-w\|_{L^2(\omega_j)} \leq Cd_j^{m+1}\|u\|_{H^{m+1}(\omega_j)}$ for a constant C independent of u, μ, and j. If $u = 0$ on $\overline{\omega}_j \cap \partial\Omega$, then we can chose $w \in \Psi_j^\mu$.*

Proof. We are especially interested in the case when $u = 0$ on $\overline{\omega}_j \cap \partial\Omega$, so we shall deal with this case in detail. The other one is proved in exactly the same way.

Let us consider $v = u \circ \tilde g_j$. Since $\tilde g_j$ maps $\mathbb{R}^{n-1} = \{x_n = 0\}$ to a surface containing $\overline{\omega}_j \cap \partial\Omega$, we obtain that $v = 0$ on $\mathbb{R}^{n-1}$. For h_μ small enough, we can assume that $\tilde g_j^{-1}(\omega_j)$ lies on one side of $\mathbb{R}^{n-1}$. Let U be the union of the closure of $\tilde g_j^{-1}(\omega_j)$ and of its symmetric subset with respect to $\mathbb{R}^{n-1}$. Define $v_1 \in H^1(U)$ to be the odd extension of v (odd with respect to the reflection about the subspace $\mathbb{R}^{n-1}$). Let p_1 be the projection of v_1 onto the subspace $\mathcal{P}_m$ of polynomials of degree m on U. This projection maps even functions to even functions and odd functions to odd functions. Hence p_1 is also odd. In particular, $p_1 = 0$ on $\mathbb{R}^{n-1}$. We also know from standard approximation results [17] that

$$\|v_1 - p_1\|_{H^1(U)} \leq Cd_j^m\|v_1\|_{H^{m+1}(U)}.$$

Then

$$\|u - p_1\circ\tilde g_j^{-1}\|_{H^1(\omega_j)} \leq C\|v_1 - p_1\|_{H^1(U)} \leq Cd_j^m\|v_1\|_{H^{m+1}(U)} \leq Cd_j^m\|u\|_{H^{m+1}(\omega_j)}.$$

Let $w = p_1\circ\tilde q_j^{-1}$. The lemma follows from

$$\begin{aligned}\|u-w\|_{H^1(\omega_j)} &\leq \|u - p_1\circ\tilde g_j^{-1}\|_{H^1(\omega_j)} + \|p_1\circ\tilde g_j^{-1} - p_1\circ\tilde q_j^{-1}\|_{H^1(\omega_j)}\\ &\leq Cd_j^m\|u\|_{H^{m+1}(\omega_j)} + Cd_j^m\|p_1\|_{H^1(\omega_j)} \leq Cd_j^m\|u\|_{H^{m+1}(\omega_j)},\end{aligned}$$

where we have used also Lemma 4.1.

To prove the relation $\|u-w\|_{L^2(\omega_j)} \leq Cd_j^{m+1}\|u\|_{H^{m+1}(\omega_j)}$, we first notice that Poincaré's inequality gives

$$\|v_1 - p_1\|_{L^2(U)} \leq Cd_j\|v_1 - p_1\|_{H^1(U)} \leq Cd_j^{m+1}\|v_1\|_{H^{m+1}(U)}.$$

The rest is exactly the same. □

We are ready now to prove Assumption 2. See [5], section 6.1, and [11] for related results.

Proposition 4.8. *The sequence of GFEM spaces S_μ satisfies Assumption 2.*

Proof. We proceed as in [11], Theorem 3.2. Let $u \in H^{m+1}(\Omega)$. If $\overline{\omega}_j$ does not intersect $\partial\Omega$, we define $w_j \in \Psi_j = \mathcal{P}_m$ to be the orthogonal projection of u onto $\mathcal{P}_m$ in $H^1(\omega_j)$. Otherwise, we define $w_j \in \Psi_j$ using Lemma 4.7. Then let $w = \sum_j \phi_j w_j$. By using Lemma 4.7, the definition of the local approximation spaces Ψ_j (Condition C), and the bounds on $\|\nabla\phi_j\|_{L^\infty}$ (Condition B), we obtain

$$
\begin{aligned}
|u - w|_{H^1(\Omega)} &= \Big|\sum_j \phi_j(u - w_j)\Big|_{H^1(\Omega)} \\
&\leq \sum_j \big(\|\phi_j\|_{L^\infty}|u - w_j|_{H^1(\omega_j)} + \|\nabla\phi_j\|_{L^\infty}\|u - w_j\|_{L^2(\omega_j)}\big) \\
&\leq \sum_j \big(C d_j^m \|u\|_{H^{m+1}(\omega_j)} + C d_j^{-1} d_j^{m+1}\|u\|_{H^{m+1}(\omega_j)}\big) \leq C\kappa h_\mu^m \|u\|_{H^{m+1}(\Omega)}.
\end{aligned}
$$

This completes the result. □

5. Non-homogeneous boundary conditions

In this section we provide an approach to the non-homogeneous Dirichlet boundary conditions. That is, consider the boundary value problem.

$$
\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{29}
$$

Our approach it to reduce again to the case $g = 0$ and then to use the results on the homogeneous Dirichlet problem (1). In a purely theoretical framework, this is achieved using an extension G of g and then solving the problem $-\Delta w = f + \Delta G$, $w = 0$ on $\partial\Omega$. The solution of (29) will then be $u = w + G$. This gives that the problem (29) has a unique solution $u \in H^{p+1}(\Omega)$ for any $f \in H^{p-1}(\Omega)$ and $g \in H^{1/2+p}(\partial\Omega)$ and it satisfies

$$
\|u\|_{H^{p+1}(\Omega)} \leq C\big(\|f\|_{H^{p-1}(\Omega)} + \|g\|_{H^{1/2+p}(\partial\Omega)}\big),
$$

for a constant $C > 0$ that depends only on Ω and $p \in \mathbb{Z}_+$. (This result is valid also for $p = 0$.)

In practice, however, we need to slightly modify this approach since it is not practical to construct the extension G (this is especially a problem if g has low regularity, that is, if g is a distribution, for instance). We will be looking therefore for a sequence G_k of *approximate extensions of* g.

The construction of such a sequence of approximate extension as well as the analysis of the resulting method are the main results of this section. Other methods for constructing G_k are certainly possible. We begin with an axiomatic approach, postulating the existence of the sequence G_k, $k \in \mathbb{N}$. Let $1 \leq p \leq m$. We consider a sequence G_k satisfying the following assumption. Recall that the spaces $\tilde{S}_k \supset S_k$ were defined in Equation (17) and are variants of the spaces S_k that are not required to satisfy, even approximately, the boundary conditions.

Assumption 3 (approximate extensions). We assume that there exists a constant $C > 0$ such that, for any $g \in H^{m+1/2}(\partial\Omega)$, there exists a sequence

$G_k \in \tilde{S}_k$ such that $\|G_k|_{\partial\Omega} - g\|_{H^{1/2}(\partial\Omega)} \leq Ch_k^m \|g\|_{H^{m+1/2}(\partial\Omega)}$ and $\|G_k\|_{H^{m+1}(\Omega)} \leq C\|g\|_{H^{m+1/2}(\partial\Omega)}$.

We now check that it is possible to choose $G_k \in \tilde{S}_k$ satisfying Assumption 3. Indeed, this is the case since Assumption 3 is satisfied if the other assumptions and conditions are satisfied. We follow the method in [4].

Proposition 5.1. *There exist continuous linear maps $I_k : H^{m+1}(\Omega) \to \tilde{S}_k$, such that*

$$|u - I_k(u)|_{H^r(\Omega)} \leq Ch_k^{m+1-r} \|u\|_{H^{m+1}(\Omega)}, \tag{30}$$

for $r = 0$ and $r = 1$.

Proof. For $u \in H^{m+1}(\Omega)$ and j fixed, let $v = u \circ \tilde{g}_j$. The Taylor polynomial of degree m of v averaged over $\tilde{g}^{-1}(\omega_j)$ is given by

$$P_j(x) := Q_j^m v(x) = \int_{\tilde{g}^{-1}(\omega_j)} Q_{y,v,n}(x)\Phi_j(y)\, dy, \tag{31}$$

where

$$Q_{y,v,n}(x) = v(y) + \sum_{i=1}^{n} \partial_i v(y)(x_i - y_i) + \ldots + \sum_{|\alpha|=m} \frac{v^{(\alpha)}(y)}{\alpha!}(x-y)^\alpha, \quad \alpha! = \alpha_1! \ldots \alpha_n!,$$

is the Taylor polynomial of v at y of degree m and $\Phi_j \in C_c^\infty(\tilde{g}^{-1}(\omega_j))$ is a function with integral 1. Then, by the Bramble–Hilbert Lemma, we have

$$|v - P_j|_{H^s(\tilde{g}^{-1}(\omega_j))} \leq Ch_k^{m+1-s} |v|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}, \quad \text{for all } 0 \leq s \leq m+1. \tag{32}$$

Consider $w_j := P_j \circ \tilde{q}_j^{-1} \in \tilde{\Psi}_j$. Let $w := \sum_j \phi_j w_j$. Then,

$$\begin{aligned} |u - w|_{H^r(\Omega)}^2 &\leq C \sum_j |\phi_j(u - w)|_{H^r(\Omega)}^2 \leq C \sum_j |\phi_j(u - w)|_{H^r(\omega_j)}^2 \\ &\leq C \sum_j \sum_{i=0}^{r} |\phi_j|_{W^{i,\infty}(\omega_j)}^2 |u - w_j|_{H^{r-i}(\omega_j)}^2 \\ &\leq C \sum_j \sum_{i=0}^{r} |\phi_j|_{W^{i,\infty}(\omega_j)}^2 [|u - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 + |P_j \circ \tilde{g}_j^{-1} - P_j \circ \tilde{q}_j^{-1}|_{H^{r-i}(\omega_j)}^2]. \end{aligned} \tag{33}$$

By changing variables and (32), we obtain

$$\begin{aligned} |u - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 &= |v \circ \tilde{g}_j^{-1} - P_j \circ \tilde{g}_j^{-1}|_{H^{r-i}(\omega_j)}^2 \leq C|v - P_j|_{H^{r-i}(\tilde{g}_j^{-1}(\omega_j))}^2 \\ &\leq Ch_k^{2(m+1-r+i)} |v|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}^2 = Ch_k^{2(m+1-r+i)} |u \circ \tilde{g}_j|_{H^{m+1}(\tilde{g}^{-1}(\omega_j))}^2 \\ &\leq Ch_k^{2(m+1-r+i)} |u|_{H^{m+1}(\omega_j)}^2. \end{aligned} \tag{34}$$

Also, from Lemma 4.1 and the definition (31) of P_j, we have

$$\begin{aligned} |P_j \circ \tilde{g}_j^{-1} - P_j \circ \tilde{q}_j^{-1}|_{H^{r-i}(\omega_j)}^2 &\leq Ch_k^{2(m+1-r+i)} \|P_j\|_{H^1(\omega_j)}^2 \\ &\leq Ch_k^{2(m+1-r+i)} \|u\|_{H^{m+1}(\omega_j)}. \end{aligned} \tag{35}$$

From (33), (34), (35), and Condition B, it follows that

(36)
$$
\begin{aligned}
|u-w|^2_{H^r(\Omega)} &\le C\sum_j\sum_{i=0}^{r} h_k^{-2i}[h_k^{2(m+1-r+i)}|u|^2_{H^{m+1}(\omega_j)} + h_k^{2(m+1-r+i)}\|u\|^2_{H^1(\omega_j)}]\\
&\le Ch_k^{2(m+1-r)}\sum_j \|u\|^2_{H^{m+1}(\omega_j)} \le C\kappa h_k^{2(m+1-r)}\|u\|^2_{H^{m+1}(\Omega)},
\end{aligned}
$$

for all $0 \le r \le m+1$.

Define $I_k(u) := w$. Clearly I_k is a linear map from $H^{m+1}(\Omega)$ to $\tilde{S}_k$ which satisfies (30). This ends the proof of the proposition. ☐

Remark 5.2. If we assume the stronger condition stated in Remark 4.3 on the m-degree polynomial q_j, then the conclusion of Proposition 5.1 is valid for $0 \le r \le m+1$ (the proof being exactly the same).

From this proposition we obtain right away the Assumption 3.

Proposition 5.3. *For any $g \in H^{m+1/2}(\partial\Omega)$ there exists a sequence $G_k \in \tilde{S}_k$ satisfying Assumption 3.*

Proof. Let us chose $G \in H^{m+1}(\Omega)$ extending g, $\|G\|_{H^{m+1}(\Omega)} \le C\|g\|_{H^{m+1/2}(\partial\Omega)}$, with C independent of g. Then choose $G_k = I_k(G)$, with I_k as in Proposition 5.1. ☐

Let w_k be the solution of

$$-\Delta w_k = f + \Delta G_k \text{ on } \Omega, \qquad w_k = 0 \text{ on } \partial\Omega. \tag{37}$$

Also, let $(w_k)_\mu \in S_\mu$ be the discrete solutions of this equation, namely, the solution of the discrete, variational problem

$$B((w_k)_\mu, v) = (f + \Delta G_k, v)_{L^2(\Omega)}, \quad v \in S_\mu, \tag{38}$$

where $f \in H^{m-1}(\Omega)$ is the data of Equation (29).

The main result of this section is the following theorem.

Theorem 5.4. *Suppose Assumptions 1 and 2 and Conditions A(h_μ), B, C, and D are satisfied. Let $u_k := (w_k)_k + G_k \in \tilde{S}_k$. Then there exists a constant $C > 0$ such that the solution $u \in H^{m+1}(\Omega)$ of Equation (29) satisfies*

$$\|u - u_k\|_{H^1(\Omega)} \le Ch_k^m\big(\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}\big).$$

Proof. We have that $v_k := w_k + G_k$ solves the boundary value problem

$$-\Delta v_k = f \text{ on } \Omega, \qquad v_k = G_k \text{ on } \partial\Omega.$$

Hence the difference $u - v_k$ solve the boundary value problem $\Delta(u - v_k) = 0$, $(u - v_k) = g - G_k$ on $\partial\Omega$. From this and Assumption 3 we obtain

$$\|u - v_k\|_{H^1(\Omega)} \le C\|g - G_k\|_{H^{1/2}(\partial\Omega)} \le Ch_k^m\|g\|_{H^{m+1/2}(\partial\Omega)}. \tag{39}$$

Theorem 1.1 and Assumption 3, which is satisfied by Proposition 5.3, then give

$$
\begin{aligned}
&\|v_k - u_k\|_{H^1(\Omega)} = \|w_k - (w_k)_k\|_{H^1(\Omega)} \le Ch_k^m\|f + \Delta G_k\|_{H^{m-1}(\Omega)}\\
&\quad\le Ch_k^m\big(\|f\|_{H^{m-1}(\Omega)} + \|G_k\|_{H^{m+1}(\Omega)}\big) \le Ch_k^m\big(\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}\big).
\end{aligned}
$$

Hence

$$\|v_k - u_k\|_{H^1(\Omega)} = \|w_k - (w_k)_k\|_{H^1(\Omega)} \le C h_k^m \big(\|f\|_{H^{m-1}(\Omega)} + \|g\|_{H^{m+1/2}(\partial\Omega)}\big). \tag{40}$$

The result follows from Equations (39) and (40). □

6. Conclusions and comments and further work

We now summarize our main results and compare them with those of Berger, Scott, and Strang [12] and Nitsche [28]. We also discuss the mixed Dirichlet-Neumann problem and suggest some further work.

6.1. Conclusions. Let S_μ, $\mu \in \mathbb{N}$, be the GFEM spaces associated to the "flat-top" data $\Sigma^\mu = \{\omega_j^\mu, \phi_j^\mu, \Psi_j^\mu, \omega_j^{*\mu}\}_{j=1}^{N_\mu}$ as in Subsection 3.2. In particular, they satisfy the conditions $\mathrm{A}(h_\mu)$, B, C, and D, for a fixed set of structural constants A, C_j, σ, and κ and $h_\mu \to 0$. (The parameters h_μ are the sizes of the patches ω_j.) We know from [11] that it is always possible to find a sequence like that, because Ω is smooth.

Let $u_\mu \in S_\mu$ be the discrete solution of the Dirichlet problem (1) (*i.e.*, $-\Delta u = f$, $u = 0$ on the boundary of the smooth, bounded domain Ω). That is, u_μ is given by Equation (2). Then we have

Theorem 6.1. *The sequence S_μ satisfies the Assumptions 1 and 2 and hence*

$$\|u - u_\mu\|_{H^1(\Omega)} \le C h_\mu^m \|u\|_{H^{m+1}(\Omega)},$$

where $C > 0$ is a constant that is independent of μ and $f \in H^{m-1}(\Omega)$.

We therefore obtain *quasi-optimal rates of convergence* for our approximate solutions $u_\mu \in S_\mu$, in the sense that the error estimate has the same order as the best approximation in the spaces S_μ.

The definition of the discrete solution u_μ is as in [12]. In that paper, Berger, Scott, and Strang obtain for $m > 3/2$ the estimate

$$\|u - u_\mu\|_{H^1(\Omega)} \le C \left(\inf_{\xi \in S_\mu} \|u - \xi\|_{H^1(\Omega)} + \|u - u_\mu\|_{H^m(\Omega)} \sup_{0 \ne v \in S_\mu} \frac{\|v\|_{H^{3/2-m}(\Omega)}}{\|v\|_{H^1(\Omega)}} \right). \tag{41}$$

Our result is thus an extension of the result of [12] to the case $m = 1$, to which the methods of that paper do not seem to apply. The case $m > 3/2$ seems not to be enough to provide optimal rates of convergence directly.

The definition of the discrete solution $u_h \in S_h$ in Nitsche's paper [28] is such that $(\Delta u_h + f, v_h) = 0$ for all $v_h \in S_h$, so it is different from ours. In addition to our approximability assumption (Assumption 1), in [28], Nitsche also requires an approximation property at the boundary and $\|u\|_{H^1(\partial\Omega)} \le C h^{-\gamma} \|u\|_{H^1(\Omega)}$ instead of our Assumption 2. These assumptions, slightly stronger than ours, also lead to optimal rates of convergence.

As in the isoparametric methods, our method uses polynomial approximations of the boundary. However, we do not have to use non-polynomial approximation functions of non-polynomial changes of coordinates.

6.2. **Mixed boundary value problems.** Let us comment now a little on the mixed boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial_D\Omega, \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial_N\Omega, \tag{42}$$

where $\partial\Omega$ is the disjoint union of the closed subsets $\partial_D\Omega$ and $\partial_N\Omega$. In particular, both $\partial_D\Omega$ and $\partial_N\Omega$ will be closed surfaces. In case $\partial_D\Omega$ is empty (the Neumann problem), we require our approximation spaces S_μ to be such that $\int_\Omega v_\mu(x)dx = 0$ for all $v_\mu \in S_\mu$.

Then our results remain the same if we consider the following variant of our Assumptions 1 and 2, in which $\partial\Omega$ was replaced with $\partial_D\Omega$.

- *Assumption 1m:* $\|v_\mu\|_{H^{1/2}(\partial_D\Omega)} \leq Ch_\mu^m \|v_\mu\|_{H^1(\Omega)}$ for any $v_\mu \in S_\mu$, and
- *Assumption 2m:* for any $u \in H^{j+1}(\Omega)$, $0 \leq j \leq m$, $u = 0$ on $\partial_D\Omega$, there exists $u_I \in S_\mu$ such that $|u - u_I|_{H^1(\Omega)} \leq Ch_\mu^j \|u\|_{H^{j+1}(\Omega)}$.

The definition of discrete solution u_μ is the same as before: $u_\mu \in \tilde{S}_\mu$ is such that

$$(\nabla(u - u_\mu), \nabla v_\mu) = 0, \quad \text{for all } v_\mu \in \tilde{S}_\mu. \tag{43}$$

An inspection of our arguments used in the proof of Proposition 4.8 shows that, if only Conditions A(h_μ) and B are satisfied, then we can take $S_\mu = \{u_\mu \in \tilde{S}_\mu, \int_\Omega u_\mu(x)dx = 0\}$. The analog of Theorem 1.1 then follows as before.

Theorem 6.2. *The (unique) solutions u and u_μ of Equations* (42) *and* (43)*, respectively, with $f \in H^{p-1}(\Omega)$ satisfy*

$$\|u - u_\mu\|_{H^1(\Omega)} \leq Ch_\mu^p \|u\|_{H^{p+1}(\Omega)} \leq Ch_\mu^p \|f\|_{H^{p-1}(\Omega)},$$

for constants independent of μ and f.

Let us observe that for the mixed Dirichlet-Neumann boundary conditions, Conditions C and D can be weakend by replacing $\partial\Omega$ with $\partial_D\Omega$. Thus, in the case of pure Neumann boundary condition case, Conditions C and D are not required. However, in the analysis of elliptic boundary-value problems with mixed Dirichlet and Neumann boundary conditions, the Conditions C and D have to be considered only for the indices j corresponding to the patches ω_j that touch the portion of the boundary on which the Dirichlet boundary conditions are imposed. See [4, 5, 11] and references therein for more information on meshless methods for Neumann boundary conditions.

6.3. **Further problems.** In spite of all these differences in assumptions and definitions between [12, 28] and our paper, the main issue seems to be providing simple examples of spaces satisfying the various assumptions used in these papers. For instance, it would be interesting to provide other examples of spaces S_μ satisfying Assumptions 1 and 2. It would also be interesting to see if a modification of the uniform partition of unity can give, by restriction, spaces S_μ satisfying these Assumptions. Finally, it would be important to integrate our results with the issues arising from numerical integration and to provide explicit numerical examples testing our results.

Some numerical tests and related theoretical results can be found, for example, in [7] and [21].

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